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The Eigenvectors of the Discrete Fourier Transform: A Version of the Hermite Functions

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The matrix appearing in the discrete Fourier transform based on N points is given by

$$(F(N))_{n,m} = N^{-1/2}\omega^{(n-1)(m-1)}, \qquad 1 \leqslant n, m \leqslant N, \quad \omega = e^{i2\pi/N}.$$

Its eigenvalues are ± 1 and $\pm i$, a fact which has been rediscovered several times. The corresponding multiplicities have also been determined. In [2, 3] one finds these multiplicities and a complete set of eigenvectors. For a variety of proofs of these facts and more references, see [1]. See also [4].

The purpose of this note is to exhibit, for each N, a tridiagonal matrix $T(N)$ with (essentially) *simple* spectrum which commutes with $F(N)$. Thus, the eigenvectors of $T(N)$ give a "natural" basis of eigenvectors of $F(N)$.

We also treat the case of the *centered* discrete Fourier transform (see Section 2). In this latter case our tridiagonal matrix can be viewed as the discrete analogue of the Hermite operator showing up in Schroedinger's equation for the harmonic oscillator. This analogy is spelled out later in the paper. In particular our tridiagonal matrix becomes, in the limit, the second-order Hermite differential operator.

In general it is *impossible* to find a tridiagonal matrix with simple spectrum in the commutator of a given matrix. When this is possible we obtain a fantastic simplification in the problem of computing eigenvectors of the given matrix: they are given by those of the tridiagonal matrix, and thus we get a (numerically) workable problem. Moreover, for theoretical purposes at least, all the machinery of orthogonal polynomials becomes relevant: The tridiagonal matrix gives rise to a set of orthogonal polynomials

$$p_i(\lambda), \qquad i = 0,\ldots, N$$

and, if λ_j denote the zeros of $p_N(\lambda)$, then the eigenvectors of the original matrix are

$$V_j = [p_0(\lambda_j), p_1(\lambda_j),\ldots, p_{N-1}(\lambda_j)]^T, \qquad 1 \leqslant j \leqslant N.$$

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Besides the instances discussed here, "naturally appearing" matrices and operators with this property show up in the work of Slepian, Landau, and Pollak (see [5] and its references). Most of these examples involve convolution-type operators or Toeplitz matrices. Some matrices of the Hankel type also enjoy this property (see [6]).

A good example of naturally appearing matrices, where the property mentioned above *fails*, is given by matrices of Toeplitz and Hankel type connected with the discrete Fourier transform and the work of I. Good, C. Rader, S. Winograd, and other people. The matrices in question can be found in [7], and it is not hard to show the failure of the property in question.

1. The Tridiagonal Matrix $T(N)$

Consider a matrix $T(N)$ of the form

$$\begin{aligned} T(N)_{m,m} &= a_m, & 1 \leqslant m \leqslant N, \\ T(N)_{m,m+1} = T(N)_{m+1,m} &= b_m, & 1 \leqslant m \leqslant N-1, \\ T(N)_{m,n} &= 0, & \text{if } |m-n| > 1. \end{aligned}$$

Assume—at a risk—that

$$a_1 = 1, \qquad a_2 = 0, \qquad b_1 = 0, \qquad b_2 = 1$$

and compute the elements $(1, m)$ and $(2, m)$ of the commutator

$$\sqrt{N}(T(N)\,F(N) - F(N)\,T(N)).$$

Setting these elements equal to zero, the conditions

$$1 - b_{m-1} - a_m - b_m = 0 \tag{1}$$

$$\omega^{2(m-1)} - \omega^{m-2} b_{m-1} - \omega^{m-1} a_m - \omega^m b_m = 0 \tag{2}$$

are obtained.

From (1) and (2)

$$b_m = \omega^{-1} b_{m-1} + (1 + \cdots + \omega^{m-2})$$

which, being a first-order nonhomogeneous difference equation for b_m, gives the solution

$$b_m = \sum_{k=2}^{m} \omega^{(k-m)} (1 + \cdots + \omega^{k-2}).$$

This expression can be manipulated a bit further to obtain

$$b_m = \frac{\sin(\pi/N)\, m \sin(\pi/N)(m-1)}{2(\sin(\pi/N))^2 \cos(\pi/N)}.$$

The expression for a_m is given by

$$\begin{aligned} a_m &= 1 - b_m - b_{m-1} \\ &= 1 - \frac{2\cos(\pi/N)(\sin(\pi/N)(m-1))^2}{2(\sin(\pi/N))^2 \cos(\pi/N)} \\ &= 1 - \left(\frac{\sin(\pi/N)(m-1)}{\sin(\pi/N)}\right)^2. \end{aligned}$$

Since $b_1 = 0$ the matrix $T(N)$ breaks into two pieces: a one-by-one and an $(N-1) \times (N-1)$ block. The large block has its off-diagonal elements all nonvanishing and thus has *simple spectrum.* $T(N)$ has $\lambda = 1$ as its only repeated eigenvalue.

Moreover,

$$a_2 = a_N, \qquad a_3 = a_{N-1}, \qquad a_4 = a_{N-2}$$

and

$$b_2 = b_{N-1}, \qquad b_3 = b_{N-2}, \qquad b_4 = b_{N-3}, \ldots.$$

Since a_m and b_m are "even" around the center location $[(N/2)]$ the eigenvectors are even or odd. To complete the proof of the relation

$$T(N)\, F(N) = F(N)\, T(N) \tag{3}$$

one has to check that the expressions for a_m and b_m found above annihilate all the entries in the commutator of $T(N)$ and $F(N)$. This is done below, where for convenience the matrix T is replaced by $2(\sin(\pi/N))^2 \cos(\pi/N)(T - I) \equiv T'$. We have

$$\begin{aligned} &\sqrt{N}(FT')_{m,n} \\ &\quad = \sin(\pi/N)(n-1) \sin(\pi/N)(n-2)\, e^{(2\pi i/N)(m-1)(n-2)} \\ &\qquad - 2\cos(\pi/N) \sin^2(\pi/N)(n-1)\, e^{(2\pi i/N)(m-1)(n-1)} \\ &\qquad + \sin(\pi/N)\, n \sin(\pi/N)(n-1)\, e^{(2\pi i/N)(m-1)n} \\ &\quad = e^{(2\pi i/N)(m-1)(n-1)} \sin(\pi/N)(n-1)[\sin(\pi/N)(n-2)\, e^{(-2\pi i/N)(m-1)} \\ &\qquad + \sin(\pi/N)\, n e^{(2\pi i/N)(m-1)} - 2\cos(\pi/N) \sin(\pi/N)(n-1)] \end{aligned}$$

$$= e^{(2\pi i/N)(m-1)(n-1)} \sin(\pi/N)(n-1)[(\cos(2\pi/N)(m-1)-1)$$
$$\times 2\cos(\pi/N)\sin(\pi/N)(n-1)$$
$$+ i\sin(2\pi/N)(m-1)\,2\sin(\pi/N)\cos(\pi/N)(n-1)]$$
$$= e^{(2\pi i/N)(m-1)(n-1)}[-4\sin^2(\pi/N)(n-1)\sin^2(\pi/N)(m-1)\cos(\pi/N)$$
$$+ i\sin(2\pi/N)(m-1)\sin(2\pi/N)(n-1)\sin(\pi/N)].$$

Since this expression is symmetric in (m, n) we see easily that

$$(T'F)_{m,n} = (FT')_{m,n}$$

and thus (3) is proved.

One can show that except for an additive multiple of I and scaling, the matrix above is the only tridiagonal matrix, with $b_{N-1} \neq 0$, which commutes with $F(N)$.

It is easy to see that the $(N-1) \times (N-1)$ block of $T(N)$ commutes with the matrix obtained by deleting the first row and column of $F(N)$. This large block of $T(N)$ has eigenvectors (with distinct eigenvalues) $v_1 = (1, 1, ..., 1)^{\mathrm{T}}$, $v_2, ..., v_{N-1}$. One obtains a "natural" set of eigenvectors for $F(N)$ by choosing

$$w_0 \equiv (1, 0, ..., 0)^{\mathrm{T}}$$
$$w_i \equiv (0, v_i)^{\mathrm{T}}, \qquad i = 1, ..., N-1.$$

2. (Centered) Discrete Fourier Transform

For convenience we look only at the case of an even number of points.

The (centered) discrete Fourier transform matrix based on $2N$ points is given by

$$F(2N)_{n,m} = (2N)^{-1/2}\,\omega^{(-N+n)(-N+m)}, \qquad 1 \leqslant n, m \leqslant 2N, \quad \omega = e^{2\pi i/2N}.$$

Consider the symmetric $2N \times 2N$ tridiagonal matrix $T(2N)$ given by

$$T(2N)_{ii} = (\sin(\pi i/2N)/\sin(\pi/2N))^2$$
$$T(2N)_{i,i+1} = (\sin(\pi(i+1)/2N)\sin(\pi i/2N))/(2\cos(\pi/2N)\sin^2(\pi/2N).$$

One easily checks that this matrix commutes with $F(2N)$, and has $\lambda = 0$ as its only repeated eigenvalue.

Our aim now is to show that if one picks scalars $\alpha(N)$ and $\beta(N)$ properly and considers

$$S(2N) \equiv \beta(N)(T(2N) + \alpha(N)\,I),$$

then $S(2N)$ approaches the differential operator

$$\frac{d^2}{dx^2} - x^2$$

acting on functions defined on R. To set the stage we recall some standard facts.

The action of $F(2N)$ on a vector $f(m)$ $m = 1,..., 2N$ is given by

$$\frac{1}{\sqrt{2N}} \sum_{m=1}^{2N} e^{(-N+m)(-N+n)2\pi i/2N} f(m) = \frac{1}{\sqrt{2N}} \sum_{k=-N+1}^{N} e^{2\pi ikl/2N} \tilde{f}(k). \quad (4)$$

Here $k = -N + m$, $l = -N + n$, and

$$\tilde{f}(k) = f(-N + m).$$

The sum above can be considered as a Riemann sum approximation to the Fourier transform of a function $\tilde{\tilde{f}}$ on R given by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \tilde{\tilde{f}}(y)\, dy. \quad (5)$$

For this we need to take

$$(\sqrt{2\pi}/\sqrt{2N})\, k \equiv \sqrt{\pi/N}(-N + m) \to y$$

$$(\sqrt{2\pi}/\sqrt{2N})\, l \equiv \sqrt{\pi/N}(-N + n) \to x. \quad (6)$$

Indeed, in this fashion the sum in (4) becomes the Riemann sum approximation, with mesh size $\sqrt{\pi/N}$ and $2N$ subintervals, to the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\sqrt{N\pi}}^{\sqrt{N\pi}} e^{ixy} \tilde{\tilde{f}}(y)\, dy.$$

For $\tilde{\tilde{f}}$ which decays at infinity fast enough, this approaches the integral (5). Observe that if L denotes the usual second difference matrix (symmetric, tridiagonal, with $L_{nn} = -2$, $L_{n,n+1} = 1$) and D denotes the diagonal matrix with

$$D_{nn} = \sin(n\pi/2N), \qquad 1 \leqslant n \leqslant 2N$$

we have

$$2 \sin^2(\pi/2N) \cos(\pi/2N)\, T(2N) = DLD + 2(1 + \cos(\pi/2N))\, D^2. \quad (7)$$

From (6) above we get

$$n \sim x\sqrt{N/\pi} + N$$

and thus

$$D_{nn} = \sin(n\pi/2N) \approx \sin(\pi/2N)(x\sqrt{N/\pi} + N) = \cos(x/2)\sqrt{\pi/N}.$$

From (7) we get

$$\frac{\pi}{2N}\left(T(2N) - \frac{8N^2}{\pi^2}\right) = \frac{\pi}{2N}\frac{DLD}{2\sin^2(\pi/2N)\cos(\pi/2N)} + \frac{\pi}{2N}\frac{2(1+\cos(\pi/2N))\,D^2}{2\sin^2(\pi/2N)\cos(\pi/2N)} - \frac{4N}{\pi}.$$

Now as $N \to \infty$ the first summand on the right converges to d^2/dx^2 (recall that the mesh size was $\sqrt{\pi/N}$). The second one behaves like

$$\frac{\pi}{N}\frac{4N^2}{\pi^2}\left(1 - \frac{x^2}{8}\frac{\pi}{N}\right)^2 \cong \frac{4N}{\pi} - x^2$$

and thus the right-hand side converges to the operator

$$\frac{d^2}{dx^2} - x^2.$$

The "convergence" observed above can be upgraded to get an honest convergence theorem in one of several senses. See [8].

As an illustration we show in the accompanying tabulation the largest five eigenvalues computed numerically for four different values of N.

It is clear—especially after peeking into the next section—that these numbers are converging as $N \to \infty$ to -1, -3, -5, -7, and -9.

We have also observed numerically that the first eigenvector is well approximated by $e^{-x^2/2}$, the second one by $xe^{-x^2/2}$, and the third one by $(x^2-1)\,e^{-x^2/2}$.

For a justification see the next section.

A word of caution: The existence of a tridiagonal matrix commuting with

	λ_1	λ_2	λ_3	λ_4	λ_5
$2N = 20$	−0.822	−2.616	−4.209	−5.610	−6.835
$2N = 100$	−0.960	−2.915	−4.823	−6.687	−8.506
$2N = 500$	−0.982	−2.962	−4.924	−6.867	−8.792
$2N = 20{,}000$	−0.998	−2.996	−4.991	−6.984	−8.974

the centered Fourier matrix does not follow automatically from the corresponding fact for the "uncentered" Fourier matrix. These two Fourier matrices are the same mapping expressed in a different basis but the concept of tridiagonal is very much *basis dependent*. For instance the matrices in [7] are also "permutations" of the $(N-1)\times(N-1)$ lower block in $F(N)$ and yet no tridiagonal matrix exists in these cases.

3. Hermite Operator

The situation described above is not too surprising in view of the relation between the Hermite operator and the Fourier transform, on the real line.

The Fourier operator

$$(Ff)(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y)\, dy$$

acts on $L^2(R)$, and has a spectrum consisting of ± 1, $\pm i$.

A second-order differential operator which commutes with F is given by

$$Tf = f'' - x^2 f.$$

Indeed, if

$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{d^n x} e^{-x^2}$$

denote the Hermite polynomials, we have the well-known relation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

If we set

$$U_n(x) \equiv e^{-x^2/2} H_n(x)$$

we obtain an orthogonal complete set in $L^2(R)$. Moreover

$$TU_n = U_n'' - x^2 U_n = -(2n+1)\, U_n$$

and

$$FU_n = i^n U_n.$$

This proves that

$$TF = FT$$

on the whole of $L^2(R)$, with some obvious care as to the domain of T.

4. A Different Limiting Operator

Here we observe that the $(N-1) \times (N-1)$ block of the matrix

$$2 \sin^2(\pi/N) \cos(\pi/N)(T(N) - I)$$

with $T(N)$ given in Section 1 can be expressed as

$$DLD + 2(1 - \cos(\pi/N))\, D^2 \tag{8}$$

with D an $(N-1) \times (N-1)$ diagonal matrix given by

$$D_{nn} = \sin(n\pi/N), \qquad n = 1, \ldots, N-1$$

and L the usual second difference.

If we now multiply (8) by N^2 and take

$$N \to \infty, \qquad n/N \to x$$

we get the operator

$$\sin \pi x \frac{d^2}{dx^2} \sin \pi x + \pi^2 x^2, \qquad 0 \leqslant x \leqslant 1$$

or equivalently for $0 \leqslant x \leqslant \pi$ we have the operator

$$\sin x \frac{d^2}{dx^2} \sin x + x^2 = \frac{d}{dx} \left(\sin^2 x \frac{d}{dx} \right).$$

To find the solutions of

$$\frac{d}{dx} \left(\sin^2 x \frac{df}{dx} \right) = \lambda f(x) \tag{9}$$

it is convenient to put

$$h(x) = f(x) \sin x$$

leading to the equation

$$h''(x) = \left(\frac{\lambda}{4 \sin^2(x/2)} + \frac{\lambda}{4 \cos^2(x/2)} - 1 \right) h(x). \tag{10}$$

One can eventually conclude that

$$f(x) = (\sin x)^{-1/2} P_{1/2}^{(1/2)-\mu}(\cos x), \tag{11}$$

where $P_\nu^\mu(x)$ is the Legendre function (see [9]) and

$$\mu = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4\lambda}.$$

A simple instance of (11) is given by $\lambda = 0$ which gives $\mu = 0, 1$. Since, except for a multiplicative scalar factor

$$P_{1/2}^{1/2}(\cos x) = \cos x/\sqrt{\sin x} \qquad \text{and} \qquad P_{1/2}^{-1/2}(\cos x) = \sqrt{\sin x}$$

we obtain the solutions to (9) with $\lambda = 0$

$$f(x) = \cos x/\sin x \qquad \text{and} \qquad f(x) = 1.$$

The geometrical significance of (10) can be gleaned from [10, p. 198].

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